LIE ALGEBRAS AND COHOMOLOGY OF CONGRUENCE SUBGROUPS

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ABSTRACT. Let R be a commutative ring that is free of rank k as an abelian group, p a prime, and $SL(n, R)$ the special linear group. We show that the Lie algebra associated to the filtration of $SL(n, R)$ by p-congruence subgroups is isomorphic to the tensor product $\mathfrak{sl}_n(R \otimes_{\mathbb{Z}} \mathbb{Z}/p) \otimes_{\mathbb{F}_p} t\mathbb{F}_p[t]$, the Lie algebra of polynomials with zero constant term and coefficients $n \times n$ traceless matrices with entries polynomials in k variables over $\mathbb{F}_p$.

We use the Lie algebra structure along with the Lyndon-Hochschild-Serre spectral sequence to compute the d^2 homology differential for certain central extensions involving quotients of p-congruence subgroups. We also use the underlying group structure to obtain several homological results. For example, we compute the first homology group of the level p-congruence subgroup for $n \geq 3$. We show that the cohomology groups of the level p^r-congruence subgroup are not finitely generated for $n = 2$ and $R = \mathbb{Z}[t]$. Finally, we show that for $n = 2$ and $R = \mathbb{Z}[i]$, the Gaussian integers, the second cohomology group of the level p^r-congruence subgroup has dimension at least two as an $\mathbb{F}_p$-vector space.

1. INTRODUCTION

The method of constructing a Lie algebra from a filtered group is classical (a thorough exposition is found in [17]), and congruence subgroups have been widely studied for many years by mathematicians such as Bass, Milnor, and Serre [1], among others. Linear groups over commutative rings admit natural filtrations by p-congruence subgroups, i.e., subgroups consisting of matrices that reduce to the identity modulo powers of p. Thus, we are able to construct an associated Lie algebra, and the structure of this Lie algebra encodes homological information, as well as information concerning the structure of certain central extensions involving quotients of congruence subgroups. It should be noted that filtrations of this type, i.e., filtrations of linear groups by p-congruence subgroups, appear in linearity problems of groups ([4] and [16]).

We start with a commutative ring R that is free of rank k as a $\mathbb{Z}$-module (unless otherwise stated, we allow k to be infinite). Let $G_n(R)$ be an $n \times n$ matrix group with coefficients in R. In this paper, assume $G_n(R) = SL(n, R)$ unless otherwise stated. Examples involving other linear groups will appear in [15]. For a prime p, there is a filtration

$$(1.1) \qquad \cdots \subseteq \Gamma(G_n(R), p^r) \subseteq \cdots \subseteq \Gamma(G_n(R), p^2) \subseteq \Gamma(G_n(R), p)$$

where

$$\Gamma(G_n(R), p^r) = \ker(G_n(R) \longrightarrow G_n(R \otimes_{\mathbb{Z}} \mathbb{Z}/p^r)).$$

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Classically, there is a Lie algebra

$$\mathrm{gr}_*(\Gamma(G_n(R), p)) = \bigoplus_{r \geq 1} \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1})$$

associated to filtration (1.1) in which the Lie bracket

$$[-, -] : \mathrm{gr}_*(\Gamma(G_n(R), p)) \otimes_{\mathbb{F}_p} \mathrm{gr}_*(\Gamma(G_n(R), p)) \longrightarrow \mathrm{gr}_*(\Gamma(G_n(R), p))$$

is induced by the group commutator map. Our main results include an explicit description of the structure of the Lie algebra $\mathrm{gr}_*(\Gamma(G_n(R), p))$ for the case $G_n(R) = SL(n, R)$, as well as the following theorem.

Theorem 1 (2.4). *Let $\mathfrak{g} = \mathfrak{sl}_n(R \otimes_{\mathbb{Z}} \mathbb{Z}/p)$ and let $\mathfrak{L}$ be the kernel of the evaluation map $\mathfrak{g} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t] \longrightarrow \mathfrak{g}$, where the map sends $t \mapsto 0$. Then*

$$\mathfrak{L} \cong \mathrm{gr}_*(\Gamma(G_n(R), p))$$

as Lie algebras.

We also explore the group structure of the filtration quotients for filtration (1.1). We make use of this when computing the structure of the Lie algebra $\mathrm{gr}_*(\Gamma(G_n(R), p))$, and also in the proof of Theorem 1. Our main results concerning the filtration quotients are as follows, where $V = \{v_i\}_{\in I}$ is a $\mathbb{Z}$-basis for R.

Theorem 2 (3.2 and 3.7). *For $n \geq 2$ and $r \geq 1$,*

$$\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}) \cong \bigoplus_{n^2-1} \mathbb{F}_p[V].$$

Moreover,

$$\Gamma(G(R), p^r)/\Gamma(G(R), p^{r+1}) \cong \mathfrak{sl}_n(\mathbb{F}_p[V]).$$

The next corollary is a direct application of this theorem and a result of Bass-Milnor-Serre [1].

Corollary 3 (3.3). *For $n \geq 3$,*

$$H_1(\Gamma(G_n(R), p); \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{F}_p[V]).$$

Finally, we use the structure of the Lie algebra $\mathrm{gr}_*(\Gamma(G_n(R), p))$ to obtain some homological information concerning congruence subgroups. The Lie algebra structure can be used to explicitly compute the d^2 homology differential in the Lyndon-Hochschild-Serre spectral sequence for certain central extensions involving quotients of p-congruence subgroups. More specifically, suppose one has a central extension

$$1 \longrightarrow \Gamma_{r+s-1}/\Gamma_{r+s} \longrightarrow \Gamma_r/\Gamma_{r+s} \longrightarrow \Gamma_r/\Gamma_{r+s-1} \longrightarrow 1,$$

where we have used the convention that $\Gamma_r = \Gamma(G_n(R), p^r)$. Furthermore, suppose that defining a map $\theta : F[x, y] \longrightarrow \Gamma_r/\Gamma_{r+s}$ induces a map of extensions

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & F[x, y] & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \\
1 & \longrightarrow & \Gamma_{r+s-1}/\Gamma_{r+s} & \longrightarrow & \Gamma_r/\Gamma_{r+s} & \longrightarrow & \Gamma_r/\Gamma_{r+s-1} & \longrightarrow & 1
\end{array}
$$

where $F[x, y]$ is the free group on two generators. In many cases, the Lie algebra structure can be used to explicitly compute d^2, thus informing on $H_1(\Gamma_r/\Gamma_{r+s})$. The main result concerning this is the following.

Theorem 4 (7.2). *Let χ be the generator of $H_2(\mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}$. Given the map of extensions θ described above,*

$$d^2(\theta_*(\chi)) = [\theta_*(x), \theta_*(y)].$$

We are also able to obtain some cohomological information by examining certain subgroups of $\Gamma(G_n(R), p^r)$. For the case $n = 2$ and $R = \mathbb{Z}[t]$, we are able to produce a subgroup $\bigoplus_{i \geq 0} \mathbb{Z} \subseteq \Gamma(G_2(R), p^r)$, from which we are able to prove the following result.

Theorem 5 (5.1). *Suppose $R = \mathbb{Z}[t]$. For all $j, r \geq 1$ and for all primes p,*

$$H^j(\Gamma(G_2(R), p^r); \mathbb{F}_p)$$

is infinitely generated.

For the case $n = 2$ and $R = \mathbb{Z}[i]$, we construct a map

$$(\mathbb{Z} \oplus \mathbb{Z}) * (\mathbb{Z} \oplus \mathbb{Z}) \longrightarrow \Gamma(G_2(R), p^r).$$

Using this, we are able to prove the following theorem. In particular, this confirms that the level p^r-congruence subgroup for $SL(2, \mathbb{Z}[i])$ is not free for any $r \geq 1$.

Theorem 6 (6.1). *Suppose $R = \mathbb{Z}[i]$. For all $r \geq 1$ and for all primes p,*

$$H^2(\Gamma(G_2(R), p^r); \mathbb{F}_p) \supseteq \mathbb{F}_p \oplus \mathbb{F}_p.$$

2. Preliminaries

We now collect the notation that will be used throughout this paper: R is a commutative ring that is free of rank k as an abelian group; $V = \{v_i\}_{i \in I}$ is a $\mathbb{Z}$-basis for R; $G_n(R)$ is an $n \times n$ matrix group with coefficients in R; p is a prime; and $\Gamma(G_n(R), p^r)$ is the kernel of the natural reduction map

$$G_n(R) \longrightarrow G_n(R \otimes_{\mathbb{Z}} \mathbb{Z}/p^r).$$

The following is a classical result (see, for example, [7]):

Theorem 2.1. *There is a Lie algebra $\mathrm{gr}_*(\Gamma(G_n(R), p))$ associated to the filtration*

(2.1) $$\cdots \subseteq \Gamma(G_n(R), p^r) \subseteq \cdots \subseteq \Gamma(G_n(R), p^2) \subseteq \Gamma(G_n(R), p)$$

in which the Lie bracket is induced by the commutator map

$$[-, -] : \Gamma(G_n(R), p^r) \times \Gamma(G_n(R), p^s) \longrightarrow \Gamma(G_n(R), p^{r+s}).$$

As one of the main results of this paper will be describing the construction and structure of this Lie algebra for the special linear group, it may be worthwhile to first consider an elementary case ($n = 2$, $R = \mathbb{Z}$) in order to explicitly see the full computation of the Lie algebra structure.

Theorem 2.2. *Suppose $n = 2$ and $R = \mathbb{Z}$, so that $G_n(R) = SL(2, \mathbb{Z})$. For $r \geq 1$, consider the following matrices:*

$$A_{12,r} = \begin{pmatrix} 1 & p^r \\ 0 & 1 \end{pmatrix} \qquad A_{21,r} = \begin{pmatrix} 1 & 0 \\ p^r & 1 \end{pmatrix} \qquad A_{11,r} = \begin{pmatrix} 1 + p^r & 0 \\ 0 & 1 - p^r \end{pmatrix}.$$

The Lie algebra $\mathrm{gr}_*(\Gamma(G_n(R), p))$ is generated (as a restricted Lie algebra over $\mathbb{F}_p$) by the three matrices $A_{12,1}$, $A_{21,1}$, and $A_{11,1}$. Furthermore, the following relations are satisfied for all $r, s \geq 1$:

(1) $[A_{11,r}, A_{12,s}] = A_{12,r+s}^2$
(2) $[A_{11,r}, A_{21,s}] = A_{21,r+s}^{-2}$
(3) $[A_{12,r}, A_{21,s}] = A_{11,r+s}$
(4) $A_{ij,r}^p = A_{ij,r+1}$.

One feature of the Lie algebra $\mathrm{gr}_*(\Gamma(G_n(R), p))$ is the fact that it is finitely generated as a restricted Lie algebra whenever $k < \infty$. This is suggested by Theorem 2.2 for the case $n = 2$, and is, in fact, true for all $n \geq 2$.

We now provide a description of the Lie algebra $\mathfrak{L}$ introduced in Theorem 1. Recall that $\mathfrak{sl}_n(\mathbb{F}_p[V])$ is the Lie algebra of $n \times n$ trace zero matrices over $\mathbb{F}_p[V]$, where the Lie bracket is defined by

$$[A, B] = AB - BA$$

for $A, B \in \mathfrak{sl}_n(\mathbb{F}_p[V])$. Let $\mathbb{F}_p[t]$ be the polynomial ring in one indeterminate over $\mathbb{F}_p$. Let I denote the kernel of the evaluation map $\mathbb{F}_p[t] \longrightarrow \mathbb{F}_p$ that sends $t \mapsto 0$. Notice that I is the $\mathbb{F}_p$-linear span of $\{t^i | i > 0\}$.

Writing $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}_p[V])$, consider the Lie algebra

$$\mathfrak{L} = \mathfrak{g} \otimes_{\mathbb{F}_p} I,$$

where the Lie bracket is defined by

$$[A \otimes t^i, B \otimes t^j] = [A, B] \otimes t^{i+j} = (AB - BA) \otimes t^{i+j}$$

for $A, B \in \mathfrak{g}$. The evaluation map $t \mapsto 0$ induces a morphism of Lie algebras

$$\mathfrak{g} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t] \longrightarrow \mathfrak{g}$$

whose kernel is exactly $\mathfrak{L}$.

Lemma 2.3. *The split short exact sequence of abelian groups*

$$0 \longrightarrow I \longrightarrow \mathbb{F}_p[t] \longrightarrow \mathbb{F}_p \longrightarrow 0$$

gives rise to a split short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{g} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t] \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Proof. The splitting $\mathbb{F}_p \longrightarrow \mathbb{F}_p[t]$ in the first short exact sequence is given by the inclusion map $x \mapsto x$.

Since the tensor product is taken over the field $\mathbb{F}_p$, the fact that

$$0 \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{g} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t] \longrightarrow \mathfrak{g} \longrightarrow 0$$

is exact is immediate. The map $\mathfrak{g} \longrightarrow \mathfrak{g} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t]$ defined by $x \mapsto x \otimes t^0$ gives the splitting. In fact, since $[x, y] \mapsto [x, y] \otimes t^0 = [x \otimes t^0, y \otimes t^0]$, the sequence is split as Lie algebras. $\square$

Theorem 2.4. *Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}_p[V])$ and let $\mathfrak{L}$ be the kernel of the evaluation map $\mathfrak{g} \otimes_{\mathbb{F}_p} \mathbb{F}_p[t] \longrightarrow \mathfrak{g}$, where the map sends $t \mapsto 0$. Then*

$$\mathfrak{L} \cong \mathrm{gr}_*(\Gamma(G_n(R), p))$$

as Lie algebras.

An immediate application of Theorem 2.4 is the computation of the Lie algebra homology of $\mathrm{gr}_*(\Gamma(G_n(R), p))$. Let $\langle t^i \rangle$ denote the $\mathbb{F}_p$-span of t^i.

Corollary 2.5. *Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}_p[V])$ as above. Then*

$$H_1(\mathrm{gr}_*(\Gamma(G_n(R), p))) = (\mathfrak{g} \otimes_{\mathbb{F}_p} \langle t \rangle) \oplus (\bigoplus_{i \geq 2} H_1(\mathfrak{g}) \otimes_{\mathbb{F}_p} \langle t^i \rangle).$$

The proof of Theorem 2.4 is included in Section 10. We first need some basic results concerning the group structure of the filtration quotients for the filtration of $\Gamma(G_n(R), p)$ defined in (1.1) and (2.1).

3. The structure of the filtration quotients for $\Gamma(G_n(R), p)$

The following theorem is due to Lee and Szczarba [14].

Theorem 3.1. *Let $G_n(R) = SL(n, \mathbb{Z})$ and suppose $n \geq 3$. There is an epimorphism*

$$\varphi : \Gamma(G_n(R), p) \longrightarrow \mathfrak{sl}_n(\mathbb{F}_p)$$

whose kernel is the commutator subgroup. Thus, $H_1(\Gamma(G_n(R), p); \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{F}_p)$.

In fact, this result can be generalized. Namely, this result can be applied to the level p^r-congruence subgroup and to coefficient rings other than $\mathbb{Z}$.

Theorem 3.2. *Suppose $n \geq 2$ and $r \geq 1$. For $G_n(R) = SL(n, R)$, there is an epimorphism*

$$\varphi_r : \Gamma(G_n(R), p^r) \longrightarrow \mathfrak{sl}_n(\mathbb{F}_p[V])$$

whose kernel is $\Gamma(G_n(R), p^{r+1})$. Thus,

$$\Gamma(G_n(R), p^r) / \Gamma(G_n(R), p^{r+1}) \cong \mathfrak{sl}_n(\mathbb{F}_p[V]).$$

The proof of this theorem will be included in Section 8. An immediate application of Theorem 3.2 is the following corollary, which uses a result of Bass-Milnor-Serre [1].

Corollary 3.3. *For $n \geq 3$, $\Gamma(G_n(R), p^2) = [\Gamma(G_n(R), p), \Gamma(G_n(R), p)]$. In particular,*

$$H_1(\Gamma(G_n(R), p); \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{F}_p[V]).$$

Proof. That $\Gamma(G_n(R), p^2) \supseteq [\Gamma(G_n(R), p), \Gamma(G_n(R), p)]$ follows from the fact that filtration (2.1) is Lie-like, as will be proved in Lemma 4.3.

Let e_{ij} denote the $n \times n$ matrix with a 1 in the (i, j) position and zeros elsewhere. Let $E(G_n(R), p^r)$ be the normal subgroup of $\Gamma(G_n(R), p^r)$ generated by matrices of the form $1 + \alpha e_{ij}$, where $i \neq j$, 1 denotes the identity matrix, $\alpha \in R$, and $\alpha \equiv 0$ mod p. By the work of Bass-Milnor-Serre [1], we have $\Gamma(G_n(R), p^r) = E(G_n(R), p^r)$ and

$$E(G_n(R), p^{r+s}) \subseteq [E(G_n(R), p^r), E(G_n(R), p^s)]$$

for $n \geq 3$. Thus,

$$\Gamma(G_n(R), p^2) \subseteq [E(G_n(R), p), E(G_n(R), p)] \subseteq [\Gamma(G_n(R), p), \Gamma(G_n(R), p)].$$

Then by Theorem 3.2, $H_1(\Gamma(G_n(R), p); \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{F}_p[V])$. $\qquad\square$

Next, we provide an alternate description of the filtration quotients that we use to describe the structure of the Lie algebra $\mathrm{gr}_*(\Gamma(G_n(R), p))$. We make use of the following fact, whose proof we include for completeness.

Lemma 3.4. *Let p be a prime and suppose $A = (a_{ij}) \in \mathrm{Mat}(n, R)$. For $r \geq 1$,*

$$\det(1 + p^r A) \equiv (1 + p^r \mathrm{trace}(A)) \mod p^{r+1}.$$

In particular, if $\det(1 + p^r A) \equiv 1 \mod p^{r+1}$, it must be the case that

$$\mathrm{trace}(A) \equiv 0 \mod p,$$

i.e., $a_{nn} \equiv -(a_{11} + \cdots + a_{n-1,n-1}) \mod p$.

Proof. The proof is by induction on n. For $n = 2$, suppose

$$1 + p^r A = \begin{pmatrix} 1 + p^r a_{11} & p^r a_{12} \\ p^r a_{21} & 1 + p^r a_{22} \end{pmatrix}$$

where $a_{ij} \in R$. Then

$$\det(1 + p^r A) = 1 + p^r(a_{11} + a_{22}) + p^{2r}(a_{11}a_{22} - a_{12}a_{21})$$
$$\equiv (1 + p^r(a_{11} + a_{22})) \mod p^{r+1}.$$

If $\det(1 + p^r A) \equiv 1 \mod p^{r+1}$, it must be the case that $\mathrm{trace}(A) \equiv 0 \mod p$. This proves the claim for the case $n = 2$.

Suppose $n - 1$ and smaller cases have been proved and $1 + p^r A \in \mathrm{Mat}(n, R)$. Let $\hat{A}_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained by removing the ith row and j^{th} column from $A = (a_{ij})$. By induction, we can write

$$\det(1 + p^r A) = (1 + p^r a_{11}) \det(\hat{A}_{11}) + p^r \sum_{j=2}^{n} (-1)^{j+1} a_{1j} \det(\hat{A}_{1j})$$
$$\equiv [(1 + p^r a_{11})(1 + p^r(a_{22} + \cdots + a_{nn}))$$
$$+ p^r \sum_{j=2}^{n} (-1)^{j+1} a_{1j} \det(\hat{A}_{1j})] \mod p^{r+1}.$$

Notice that each of $\hat{A}_{1j}$ for $2 \leq j \leq n$ is an $(n-1) \times (n-1)$ matrix whose first column is

$$\begin{pmatrix} p^r a_{21} \\ p^r a_{31} \\ \vdots \\ p^r a_{nn} \end{pmatrix}.$$

If we use the cofactor expansion on this column to compute the determinant of $\hat{A}_{1j}$, it's clear that $\det(\hat{A}_{1j}) \equiv 0 \mod p^r$ for $2 \leq j \leq n$. Using this above, we see that

$$\det(1 + p^r A) \equiv (1 + p^r \mathrm{trace}(A)) \mod p^{r+1}.$$

If $\det(1 + p^r A) \equiv 1 \mod p^{r+1}$, it must be the case that $\mathrm{trace}(A) \equiv 0 \mod p$. This completes the proof of the Lemma. $\qquad\square$

Theorem 3.5. *For $r \geq s \geq 1$,*

$$\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+s}) \cong \bigoplus_{n^2-1} \mathbb{Z}/p^s\mathbb{Z}[V].$$

The proof is by induction on s. For the sake of continuity, we prove the case $s = 1$ in the next lemma, and complete the induction in Section 9. We first state a corollary that follows directly from Theorem 3.5.

Corollary 3.6. *For all primes p,*

$$\varprojlim_s \Gamma(G_n(R), p^s)/\Gamma(G_n(R), p^{2s}) = \bigoplus_{n^2-1} \hat{\mathbb{Z}}_p[V],$$

where $\hat{\mathbb{Z}}_p$ denotes the p-adic integers.

Proof. Follows immediately from Theorem 3.5 and the fact that inverse limits commute with direct sums. □

Lemma 3.7. *For $r \geq 1$,*

$$\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}) \cong \bigoplus_{n^2-1} \mathbb{F}_p[V],$$

where V is a $\mathbb{Z}$-basis for R.

Proof. Consider the following commutative diagram

$$
\begin{array}{ccccc}
\Gamma(G_n(R), p^{r+1}) & \xrightarrow{\ Id\ } & \Gamma(G_n(R), p^{r+1}) & \longrightarrow & \{1\} \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(G_n(R), p^r) & \longrightarrow & SL(n, R) & \xrightarrow{\ \phi_r\ } & SL(n, R \otimes_{\mathbb{Z}} \mathbb{Z}/p^r) \\
\downarrow & & \downarrow{\scriptstyle \phi_{r+1}} & & \downarrow{\scriptstyle Id} \\
\ker\theta_r & \longrightarrow & SL(n, R \otimes_{\mathbb{Z}} \mathbb{Z}/p^{r+1}) & \xrightarrow{\ \theta_r\ } & SL(n, R \otimes_{\mathbb{Z}} \mathbb{Z}/p^r)
\end{array}
$$

where ϕ_r, ϕ_{r+1}, and θ_r are the natural reduction maps and $\phi_r = \theta_r \circ \phi_{r+1}$. In [18], it is shown that reduction maps such as ϕ_r, ϕ_{r+1}, and θ_r are surjections. Given this, it's immediate that the three rows and two right columns are exact. A standard "diagram chase" can be used to show that the first column is exact. This gives

$$\ker\theta_r \cong \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}).$$

The reader should note that this isomorphism allows us to view

$$\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}) \subseteq SL(n, R \otimes_{\mathbb{Z}} \mathbb{Z}/p^{r+1}).$$

This will prove to be a useful fact later in this paper.

A priori, an element of $\ker\theta_r$ is a matrix of the form $1 + p^r A$ where $A = (a_{ij}) \in \mathrm{Mat}(n, R)$ and $\det(1 + p^r A) \equiv 1 \mod p^{r+1}$. Since $1 + p^r A \in SL(n, R \otimes_{\mathbb{Z}} \mathbb{Z}/p^{r+1})$, we can assume $A \in \mathrm{Mat}(n, R \otimes_{\mathbb{Z}} \mathbb{Z}/p)$.

Define a map $\Phi : \bigoplus_{n^2-1} \mathbb{F}_p[V] \longrightarrow \ker\theta_r$ by

$$(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{n,n-1}) \mapsto 1 + p^r(a_{ij})$$

where $a_{nn} = -(a_{11} + a_{22} + \cdots + a_{n-1,n-1})$. By Lemma 3.4, we see immediately that Φ is surjective. By inspection, the kernel is trivial, so that Φ is a bijection.

To check that Φ is a homomorphism, notice that

$$(1 + p^r A)(1 + p^r B) = 1 + p^r(A + B) + p^{2r} AB$$
$$\equiv 1 + p^r(A + B) \mod p^{r+1}.$$

Thus, Φ is the required isomorphism. This completes the proof. $\qquad\square$

This lemma has several immediate consequences. Recall that $V = \{v_i\}_{i \in I}$ is a $\mathbb{Z}$-basis for R. For $r \geq 1$, $1 \leq i, j \leq n$, and $i + j < 2n$, define a family of matrices by

$$A_{ij,k,r} = \begin{cases} 1 + p^r v_k e_{ij} & \text{if } i \neq j \\ 1 + p^r v_k (e_{ii} - e_{nn}) & \text{if } i = j, \end{cases}$$

where the matrices $\{e_{ij}\}$ are as defined in the proof of Corollary 3.3.

Corollary 3.8. *For* $r \geq 1$, $\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1})$ *is generated by the family of matrices* $\{A_{ij,k,r}\}$ *defined above.*

Remark. It's clear that $\det(A_{ij,k,r}) = 1$ for $i \neq j$. When $i = j$, notice that

$$\det(A_{ii,k,r}) = (1 + p^r v_k)(1 - p^r v_k) = 1 - p^{2r} v_k^2 \equiv 1 \mod p^{r+1}.$$

Since $\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}) \subseteq SL(n, R/p^{r+1}R)$, this suffices.

4. The Lie algebra associated to $SL(n, R)$

We begin with some preliminary definitions concerning the existing of the associated Lie algebra, and then describe its structure completely. The following definition is stated as in [4].

Definition 4.1. *A filtration* $\{F_n(G)\}_{n \geq 0}$ *of the group* G *is said to be Lie-like if, for all* $m, n \geq 0$, *the commutator map*

$$[-,-] : G \times G \longrightarrow G$$

given by $[g, h] \mapsto g^{-1} h^{-1} gh$ *restricts to a map*

$$[-,-] : F_m(G) \times F_n(G) \longrightarrow F_{m+n}(G).$$

The next theorem is classical and can be found in [17], for example. The theorem gives a general procedure for constructing a Lie algebra from a filtered group whenever the filtration is Lie-like.

Theorem 4.2. *Given a Lie-like filtration for the group* G, *there is an associated Lie algebra*

$$\mathrm{gr}_*(G) = \bigoplus_{n \geq 1} F_n(G)/F_{n+1}(G).$$

The Lie bracket is obtained by linearly extending the maps

$$[-,-] : F_n(G)/F_{n+1}(G) \times F_m(G)/F_{m+1}(G) \longrightarrow F_{m+n}(G)/F_{m+n+1}(G)$$

that are induced by the restriction of the commutator map.

Remark. Note that these Lie algebras are not, in general, graded Lie algebras. In other words, it's not necessarily true that $[x, y] = (-1)^{|x||y|}[y, x]$, and the graded Jacobi identity need not be satisfied.

Next we verify that the filtration of $\Gamma(G_n(R), p)$ given by (2.1) is, in fact, Lie-like.

Lemma 4.3. *Suppose that* $X \in \Gamma(G_n(R), p^r)$ *and* $Y \in \Gamma(G_n(R), p^s)$. *Then* $[X, Y] \in \Gamma(G_n(R), p^{r+s})$, *where* $[X, Y] = X^{-1}Y^{-1}XY$ *is the group commutator.*

Proof. We can write $X = 1 + p^r A$ and $Y = 1 + p^s B$ for some matrices $A, B \in \mathrm{Mat}(n, R)$. Then

$$[1 + p^r A, 1 + p^s B] = (1 + p^r A)^{-1}(1 + p^s B)^{-1}(1 + p^r A)(1 + p^s B)$$

$$= \sum_{i=0}^{\infty}(-p^r A)^i \sum_{j=0}^{\infty}(-p^s B)^j (1 + p^r A)(1 + p^s B)$$

$$= 1 + p^{r+s}(AB - BA) + p^{r+s+1}C$$

where $C \in \mathrm{Mat}(n, R)$ is some combination of A and B. Thus,

$$[X, Y] \in \Gamma(G_n(R), p^{r+s}),$$

as desired. $\qquad\square$

By Theorem 4.2, we have an associated Lie algebra

$$\mathrm{gr}_*(\Gamma(G(R), p)) = \bigoplus_{r \geq 1} \Gamma_n(G(R), p^r)/\Gamma(G_n(R), p^{r+1})$$

in which the Lie bracket is induced by the restriction of the commutator map. In the next corollary, we use the notation $\Gamma_r = \Gamma(G_n(R), p^r)$.

Corollary 4.4. *The commutator map*

$$[-, -] : \Gamma(G_n(R), p^r) \times \Gamma(G_n(R), p^s) \longrightarrow \Gamma(G_n(R), p^{r+s})$$

extends to a well-defined map on the filtration quotients

$$[-, -] : \Gamma_r/\Gamma_{r+1} \times \Gamma_s/\Gamma_{s+1} \longrightarrow \Gamma_{r+s}/\Gamma_{r+s+1}.$$

The Lie bracket

$$[-, -] : \mathrm{gr}_*(\Gamma(G_n(R), p)) \otimes_{\mathbb{F}_p} \mathrm{gr}_*(\Gamma(G_n(R), p)) \longrightarrow \mathrm{gr}_*(\Gamma(G_n(R), p))$$

is obtained by extending these maps linearly.

Proof. This follows immediately from the proof of Lemma 4.3 and [17]. $\qquad\square$

The next result will be used to describe $\mathrm{gr}_*(\Gamma(G_n(R), p))$ as a restricted Lie algebra. Recall that $\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1})$ is generated by the family of matrices $\{A_{ij,k,r}\}$ defined in Corollary 3.8.

Proposition 4.5. *The Frobenius map*

$$\Gamma(G_n(R), p^r) \longrightarrow \Gamma(G_n(R), p^{r+1})$$

defined by $A_{ij,k,r} \mapsto (A_{ij,k,r})^p$ *induces a map*

$$\psi_r^p : \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}) \longrightarrow \Gamma(G_n(R), p^{r+1})/\Gamma(G_n(R), p^{r+2}).$$

This map is an isomorphism, except in the case $p = 2$, $r = 1$. *Furthermore,* $\psi_r^p(A_{ij,k,r}) = A_{ij,k,r+1}$.

Proof. Assume that $p \neq 2$ or $r > 1$.

A typical element of $\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1})$ is of the form $1 + p^r A$ where $A \in \text{Mat}(n, R)$. In the proof of Lemma 3.7, it was noted that

$$\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}) \subseteq SL(n, R/p^{r+1}R),$$

so we can assume that $A \in \text{Mat}(n, R/pR)$. To see that the image of ψ_r^p lies in $\Gamma(G_n(R), p^{r+1})/\Gamma(G_n(R), p^{r+2})$, notice that

$$\psi_r^p(1 + p^r A) = (1 + p^r A)^p$$

$$= \sum_{i=0}^{p} \binom{p}{i} (p^r A)^i$$

$$\equiv (1 + p^{r+1} A) \mod p^{r+2}.$$

To check that ψ_r^p is a homomorphism, consider two elements $1 + p^r A$, $1 + p^r B \in \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1})$. Then

$$\psi_r^p((1 + p^r A)(1 + p^r B)) = \psi_r^p(1 + p^r(A + B))$$

$$= (1 + p^r(A + B))^p$$

$$\equiv (1 + p^{r+1}(A + B)) \mod p^{r+2}$$

$$\equiv (1 + p^{r+1} A)(1 + p^{r+1} B) \mod p^{r+2}$$

$$= \psi_r^p(1 + p^r A)\psi_r^p(1 + p^r B).$$

Thus, ψ_r^p is a homomorphism.

To see that ψ_r^p is surjective, we show that it maps onto the generators of $\Gamma(G_n(R), p^{r+1})/\Gamma(G_n(R), p^{r+2})$. For $i \neq j$,

$$(A_{ij,k,r})^p = (1 + p^r v_k e_{ij})^p$$

$$= \sum_{l=0}^{p} \binom{p}{l} (p^r v_k e_{ij})^l$$

$$\equiv (1 + p^{r+1} v_k e_{ij}) \mod p^{r+2}$$

$$= A_{ij,k,r+1}.$$

For $i = j$, a similar computation gives the following:

$$(A_{ii,k,r})^p = (1 + p^r v_k(e_{ii} - e_{nn}))^p$$

$$= \sum_{l=0}^{p} \binom{p}{l} (p^r v_k(e_{ii} - e_{nn}))^l$$

$$\equiv (1 + p^{r+1} v_k(e_{ii} - e_{nn})) \mod p^{r+2}$$

$$= A_{ii,k,r+1}.$$

Thus, $\psi_r^p(A_{ij,k,r}) = A_{ij,k,r+1}$ and ψ_r^p is surjective.

Finally, suppose that $\psi_r^p(1 + p^r A) = 1$. Then it must be the case that

$$p^{r+1} A \equiv 0 \mod p^{r+2}.$$

Since $A \in \text{Mat}(n, R/pR)$, we conclude that $A \equiv 0 \mod p$. Hence, $(1 + p^r A) \equiv 1 \mod p^{r+1}$ and ψ_r^p is injective. This completes the proof of the proposition. $\square$

We are now ready to give the structure of the Lie algebra $\mathrm{gr}_*(\Gamma(G_n(R),p))$, and we do so in the next theorem. In our computations, we make use of the following well-known results.

Lemma 4.6. *For the matrices $\{e_{ij}\}$ defined in the proof of Corollary 3.3, we have the following.*

$$[e_{ij}, e_{kl}] = e_{ij}e_{kl} - e_{kl}e_{ij} = \begin{cases} -e_{kj} & \text{if } i = l \text{ and } j \neq k \\ e_{il} & \text{if } j = k \text{ and } i \neq l \\ e_{il} - e_{kj} & \text{if } i = l \text{ and } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It's clear that the first term, $e_{ij}e_{kl}$, is non-zero only when $j = k$ and the second term, $e_{kl}e_{ij}$, is non-zero precisely when $i = l$. In these cases, $e_{ij}e_{kl} = e_{il}$ and $e_{kl}e_{ij} = e_{kj}$. $\qquad\square$

In the next theorem, we use the convention that $v_{q_1 q_2}$ denotes the product $v_{q_1} v_{q_2}$.

Theorem 4.7. *For the Lie algebra*

$$\mathrm{gr}_*(\Gamma(G_n(R),p)) = \bigoplus_{r \geq 1} \Gamma(G_n(R),p^r)/\Gamma(G_n(R),p^{r+1}),$$

the following properties are satisfied.

(1) *For all $r \geq 1$, $\Gamma(G_n(R),p^r)/\Gamma(G_n(R),p^{r+1}) \cong \bigoplus_{n^2-1} \mathbb{F}_p[V]$.*
(2) *For all $r \geq 1$, $\Gamma(G_n(R),p^r)/\Gamma(G_n(R),p^{r+1}) \cong \mathfrak{sl}_n(\mathbb{F}_p[V])$.*
(3) *For all $r \geq 1$, $\psi_r^p(A_{ij,k,r}) = A_{ij,k,r+1}$, where the matrices $\{A_{ij,k,r}\}$ are as defined in Corollary 3.8.*
(4) *$\mathrm{gr}_*(\Gamma(G_n(R),p))$ is generated as a restricted Lie algebra by the matrices $\{A_{ij,k,1}\}$.*
(5) *The Lie bracket is defined on the generators $\{A_{ij,k,r}\}$ as follows:*

$$[A_{ij,q_1,r}, A_{kl,q_2,s}] = \begin{cases} A_{kj,q_1 q_2,r+s}^{-1} & \text{if } i = l, \, j \neq k, \, i \neq j, \, k \neq l \\ A_{ii,q_1 q_2,r+s} A_{jj,q_1 q_2,r+s}^{-1} & \text{if } i = l, \, j = k, \, i \neq j, \, k \neq l \\ A_{il,q_1 q_2,r+s} & \text{if } i \neq l, \, j = k, \, i \neq j, \, k \neq l \\ A_{in,q_1 q_2,r+s}^{-2} & \text{if } i = k = l, \, i \neq j, \, j = n \\ A_{ij,q_1 q_2,r+s}^{-1} & \text{if } i = k = l, \, i \neq j, \, j \neq n, \, i \neq n \\ A_{nj,q_1 q_2,r+s}^{2} & \text{if } j = k = l, \, i \neq j, \, i = n \\ A_{ij,q_1 q_2,r+s} & \text{if } j = k = l, \, i \neq j, \, i \neq n, \, j \neq n \\ A_{in,q_1 q_2,r+s}^{-1} & \text{if } k = l, \, i \neq k, \, j \neq k, \, i \neq j, \, j = n \\ A_{nj,q_1 q_2,r+s} & \text{if } k = l, \, i \neq k, \, j \neq k, \, i \neq j, \, i = n \\ A_{ni,q_1 q_2,r+s}^{-2} & \text{if } i = j = l, \, k \neq l, \, k = n \\ A_{ki,q_1 q_2,r+s}^{-1} & \text{if } i = j = l, \, k \neq l, \, k \neq n, \, l \neq n \\ A_{in,q_1 q_2,r+s}^{2} & \text{if } i = j = k, \, k \neq l, \, l = n \\ A_{il,q_1 q_2,r+s} & \text{if } i = j = k, \, k \neq l, \, k \neq n, \, l \neq n \\ A_{nl,q_1 q_2,r+s}^{-1} & \text{if } i = j, \, k \neq l, \, i \neq l, \, i \neq k, \, k = n \\ A_{kn,q_1 q_2,r+s} & \text{if } i = j, \, k \neq l, \, i \neq l, \, i \neq k, \, l = n \\ 1 & \text{otherwise.} \end{cases}$$

Proof. The first four parts of the theorem follow immediately from Lemma 3.7, Theorem 3.2, Proposition 4.5, and Corollary 3.8, respectively. By Lemma 4.3, we

have the following:

$$[A_{ij,q_1,r}, A_{kl,q_2,s}] = \begin{cases} 1 + p^{r+s}v_{q_1}v_{q_2}[e_{ij}, e_{kl}] & \text{if } i \neq j, \ k \neq l \\ 1 + p^{r+s}v_{q_1}v_{q_2}([e_{ij}, e_{kk}] + [e_{nn}, e_{ij}]) & \text{if } i \neq j, \ k = l \\ 1 + p^{r+s}v_{q_1}v_{q_2}([e_{ii}, e_{kl}] + [e_{kl}, e_{nn}]) & \text{if } i = j, \ k \neq l \\ 1 + p^{r+s}v_{q_1}v_{q_2}([e_{ii}, e_{kk}] + [e_{nn}, e_{ii}] + [e_{kk}, e_{nn}]) & i = j, \ k = l. \end{cases}$$

Using Lemma 4.6, the reader can verify the relations listed in the statement of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Cohomology of congruence subgroups for $SL(2, \mathbb{Z}[t])$

Suppose $n = 2$ and $R = \mathbb{Z}[t]$, so that $G_n(R) = SL(n, R)$, where $\mathbb{Z}[t]$ is a polynomial ring in one variable. For fixed $r \geq 1$ and $i \geq 0$, consider the matrix

$$A_{12,i,r} = \begin{pmatrix} 1 & p^r t^i \\ 0 & 1 \end{pmatrix} \in \Gamma(G_2(R), p^r).$$

The collection $\{A_{12,i,r}\}_{i\geq 0}$ is a family of pairwise commutative matrices, each of infinite order in $\Gamma(G_2(R), p^r)$. This collection generates a copy of $\bigoplus_{i\geq 0} \mathbb{Z} \subseteq \Gamma(G_2(R), p^r)$. Thus, we have the following diagram

where i is the inclusion map, π is the natural quotient map, Φ_2^{-1} is the projection onto the second coordinate under the isomorphism Φ^{-1} defined in the proof of Lemma 3.7, and f is the composite. Notice that $f(A_{12,i,r}) = t^i$ under this mapping. We now consider the cases $p = 2$ and $p > 2$ separately.

$\underline{p = 2.}$ Recall that

$$H^*(\mathbb{F}_2[t]; \mathbb{F}_2) \cong H^*(\oplus_{i\geq 0}\mathbb{F}_2; \mathbb{F}_2) \cong \mathbb{F}_2[x_0, x_1, \ldots],$$

a polynomial ring in an infinite number of indeterminates of degree 1, where each x_i is dual to the image of $A_{12,i,r}$ in $\Gamma(G_2(R), 2^r)/\Gamma(G_2(R), 2^{r+1})$. Also,

$$H^*(\oplus_{i\geq 0}\mathbb{Z}; \mathbb{F}_2) \cong \Lambda^*(y_0, y_1, \ldots; \mathbb{F}_2),$$

an exterior algebra in an infinite number of indeterminates of degree 1 over $\mathbb{F}_2$, where each y_i is dual to $A_{12,i,r}$ in $\Gamma(G_2(R), 2^r)$.

There are induced maps in cohomology

for which $f^*(x_i) = y_i$ for $j = 1$. Thus, we see that f^* is an epimorphism for $j \geq 1$, forcing i^* to be an epimorphism for $j \geq 1$ as well. In this case, $H^j(\bigoplus_{i\geq 0} \mathbb{Z}; \mathbb{F}_2)$ is a quotient of $H^j(\Gamma(G_2(R), 2^r); \mathbb{F}_2)$. Since $H^j(\bigoplus_{i\geq 0} \mathbb{Z}; \mathbb{F}_2)$ is not finitely generated for $j \geq 1$, neither is $H^j(\Gamma(G_2(R), 2^r); \mathbb{F}_2)$.

$p > 2$. Recall that

$$H^*(\mathbb{F}_p[t]; \mathbb{F}_p) \cong H^*(\oplus_{i \geq 0} \mathbb{F}_p; \mathbb{F}_p) \cong \Lambda^*(x_0, x_1, \ldots; \mathbb{F}_p) \otimes \mathbb{F}_p[\beta x_0, \beta x_1, \ldots],$$

the tensor product of an exterior algebra with a polynomial ring. Here, each x_i has degree 1 and $\beta : H^1(\mathbb{F}_p; \mathbb{F}_p) \longrightarrow H^2(\mathbb{F}_p; \mathbb{F}_p)$ is the Bockstein homomorphism. Similar to above, each x_i is dual to the image of $A_{12,i,r}$ in $\Gamma(G_2(R), p^r)/\Gamma(G_2(R), p^{r+1})$. Also,

$$H^*(\oplus_{i \geq 0} \mathbb{Z}; \mathbb{F}_p) \cong \Lambda^*(y_0, y_1, \ldots; \mathbb{F}_p),$$

an exterior algebra in an infinite number of indeterminates of degree 1 over $\mathbb{F}_p$, where each y_i is dual to $A_{12,i,r}$ in $\Gamma(G_2(R), p^r)$.

There are induced maps in cohomology

$$H^j(\mathbb{F}_p[t]; \mathbb{F}_p) \xrightarrow{(\Phi_2^{-1} \circ \pi)^*} H^j(\Gamma(G_2(R), p^r); \mathbb{F}_p) \xrightarrow{i^*} H^j(\oplus_{i \geq 0} \mathbb{Z}; \mathbb{F}_p)$$
$$\xrightarrow{f^*} H^j(\oplus_{i \geq 0} \mathbb{Z}; \mathbb{F}_p)$$

for which $f^*(x_i) = y_i$ for $j = 1$. Thus, we see that f^* is an epimorphism for $j \geq 1$, forcing i^* to be an epimorphism for $j \geq 1$ as well. In this case, $H^j(\bigoplus_{i \geq 0} \mathbb{Z}; \mathbb{F}_p)$ is a quotient of $H^j(\Gamma(G_2(R), p^r); \mathbb{F}_p)$. Since $H^j(\bigoplus_{i \geq 0} \mathbb{Z}; \mathbb{F}_p)$ is not finitely generated for $j \geq 1$, neither is $H^j(\Gamma(G_2(R), p^r); \mathbb{F}_p)$. Thus, we have proved the following theorem.

Theorem 5.1. *Suppose $R = \mathbb{Z}[t]$. For all $j, r \geq 1$ and for all primes p,*

$$H^j(\Gamma(G_2(R), p^r); \mathbb{F}_p)$$

is infinitely generated.

6. Cohomology of congruence subgroups for $SL(2, \mathbb{Z}[i])$

Suppose $n = 2$ and $R = \mathbb{Z}[i]$, so that $G_n(R) = SL(2, \mathbb{Z}[i])$, where $\mathbb{Z}[i]$ is the Gaussian integers. Let e_{ij} denote the 2×2 matrix with a 1 in the (i, j) position and zeros elsewhere. For fixed $r \geq 1$, $\epsilon \in \{0, 1\}$, and $i \neq j$, consider the matrices $A_{ij,\epsilon,r} = 1 + p^r i^\epsilon e_{ij} \in \Gamma(G_2(R), p^r)$. Each of these four matrices has infinite order in $\Gamma(G_2(R), p^r)$. Notice that $A_{12,0,r}$ and $A_{12,1,r}$ commute, so they generate a copy of $\mathbb{Z} \oplus \mathbb{Z} \subseteq \Gamma(G_2(R), p^r)$. The same is true for the matrices $A_{21,0,r}$ and $A_{21,1,r}$. Thus, we have embeddings $i_1 : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \Gamma(G_2(R), p^r)$ and $i_2 : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \Gamma(G_2(R), p^r)$. These embeddings give

$$(\mathbb{Z} \oplus \mathbb{Z}) * (\mathbb{Z} \oplus \mathbb{Z}) \xrightarrow{i_1 * i_2} \Gamma(G_2(R), p^r) \xrightarrow{\pi} \Gamma(G_2(R), p^r)/\Gamma(G_2(R), p^{r+1})$$
$$\xrightarrow{f} \oplus_6 \mathbb{F}_p \quad \downarrow \Phi^{-1}$$

where π is the natural quotient map, Φ is the isomorphism defined in the proof of Lemma 3.7, and f is the composite. We now consider the cases $p = 2$ and $p > 2$ separately.

$p = 2$. Recall that

$$H^*(\oplus_6 \mathbb{F}_2; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, y_1, y_2, z_1, z_2],$$

a polynomial ring in six indeterminates of degree 1, where x_1 is dual to the image of $A_{12,0,r}$, x_2 is dual to the image of $A_{12,1,r}$, y_1 is dual to the image of $A_{21,0,r}$, and y_2 is dual to the image of $A_{21,1,r}$ in $\Gamma(G_2(R),2^r)/\Gamma(G_2(R),2^{r+1})$. Also,

$$H^*((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_2) \cong \Lambda^*(u_1,u_2;\mathbb{F}_2)\oplus\Lambda^*(w_1,w_2;\mathbb{F}_2),$$

where u_1 is dual to $A_{12,0,r}$, u_2 is dual to $A_{12,1,r}$, w_1 is dual to $A_{21,0,r}$, and w_2 is dual to $A_{21,1,r}$ in $\Gamma(G_2(R),2^r)$. In particular, notice that

$$H^2((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_2) \cong \mathbb{F}_2\oplus\mathbb{F}_2,$$

generated by the classes u_1u_2 and w_1w_2.

There are induced maps in cohomology

$$H^j(\oplus_6\mathbb{F}_2;\mathbb{F}_2) \xrightarrow{(\Phi^{-1}\circ\pi)^*} H^j(\Gamma(G_2(R),2^r);\mathbb{F}_2) \xrightarrow{(i_1*i_2)^*} H^j((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_2)$$

$$\Big\| $$

$$f^* \qquad\qquad H^j((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_2)$$

for which $f^*(x_i) = u_i$ and $f^*(y_i) = w_i$ for $j = 1$. Thus, we see that f^* is an epimorphism for $j = 2$, forcing $(i_1 * i_2)^*$ to be an epimorphism for $j = 2$ as well. Thus, $H^2((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_2)$ is a quotient of $H^2(\Gamma(G_2(R),2^r);\mathbb{F}_2)$, meaning $\mathbb{F}_2\oplus\mathbb{F}_2 \subseteq H^2(\Gamma(G_2(R),2^r);\mathbb{F}_2)$.

$\underline{p > 2}$. Recall that

$$H^*(\oplus_6\mathbb{F}_p;\mathbb{F}_p) \cong \Lambda^*(x_1,x_2,y_1,y_2,z_1,z_2;\mathbb{F}_2)\otimes\mathbb{F}_p[\beta x_1,\beta x_2,\beta y_1,\beta y_2,\beta z_1,\beta z_2],$$

the tensor product of an exterior algebra with a polynomial ring. Here, each x_i, y_i, and z_i is of degree 1 and $\beta : H^1(\mathbb{F}_p;\mathbb{F}_p) \longrightarrow H^2(\mathbb{F}_p;\mathbb{F}_p)$ is the Bockstein homomorphism. Similar to above, x_1 is dual to the image of $A_{12,0,r}$, x_2 is dual to the image of $A_{12,1,r}$, y_1 is dual to the image of $A_{21,0,r}$, and y_2 is dual to the image of $A_{21,1,r}$ in $\Gamma(G_2(R),p^r)/\Gamma(G_2(R),p^{r+1})$. Also,

$$H^*((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_p) \cong \Lambda^*(u_1,u_2;\mathbb{F}_p)\oplus\Lambda^*(w_1,w_2;\mathbb{F}_p),$$

where u_1 is dual to $A_{12,0,r}$, u_2 is dual to $A_{12,1,r}$, w_1 is dual to $A_{21,0,r}$, and w_2 is dual to $A_{21,1,r}$ in $\Gamma(G_2(R),p^r)$. In particular, notice that

$$H^2((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_p) \cong \mathbb{F}_p\oplus\mathbb{F}_p,$$

generated by the classes u_1u_2 and w_1w_2.

There are induced maps in cohomology

$$H^j(\oplus_6\mathbb{F}_p;\mathbb{F}_p) \xrightarrow{(\Phi^{-1}\circ\pi)^*} H^j(\Gamma(G_2(R),p^r);\mathbb{F}_p) \xrightarrow{(i_1*i_2)^*} H^j((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_p)$$

$$\Big\| $$

$$f^* \qquad\qquad H^j((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_p)$$

for which $f^*(x_i) = u_i$ and $f^*(y_i) = w_i$ for $j = 1$. Thus, we see that f^* is an epimorphism for $j = 2$, forcing $(i_1 * i_2)^*$ to be an epimorphism for $j = 2$ as well. Thus, $H^2((\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z});\mathbb{F}_p)$ is a quotient of $H^2(\Gamma(G_2(R),p^r);\mathbb{F}_p)$, meaning $\mathbb{F}_p\oplus\mathbb{F}_p \subseteq H^2(\Gamma(G_2(R),p^r);\mathbb{F}_p)$. Thus, we have proved the following theorem.

Theorem 6.1. *Suppose $R = \mathbb{Z}[i]$. For all $r \geq 1$ and for all primes p,*

$$H^2(\Gamma(G_2(R),p^r);\mathbb{F}_p) \supseteq \mathbb{F}_p\oplus\mathbb{F}_p.$$

7. Computing d^2 for central extensions of congruence subgroups

In this section, we use the convention that $\Gamma_r = \Gamma(G_n(R), p^r)$. Let $F[x, y]$ be the free group on two letters, and let K denote the kernel of the abelianization map $F[x, y] \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$. Specifying elements $\theta(x), \theta(y) \in \Gamma_r/\Gamma_{r+s}$ defines a unique homomorphism $\theta : F[x, y] \longrightarrow \Gamma_r/\Gamma_{r+s}$. We are interested in situations for which the extension

$$(7.1) \qquad 1 \longrightarrow \Gamma_{r+s-l}/\Gamma_{r+s} \longrightarrow \Gamma_r/\Gamma_{r+s} \longrightarrow \Gamma_r/\Gamma_{r+s-l} \longrightarrow 1$$

is central, and defining a map θ as above induces a morphism of extensions

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & F[x, y] & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \\
1 & \longrightarrow & \Gamma_{r+s-l}/\Gamma_{r+s} & \longrightarrow & \Gamma_r/\Gamma_{r+s} & \longrightarrow & \Gamma_r/\Gamma_{r+s-l} & \longrightarrow & 1.
\end{array}
$$

It's straightforward to determine when the first condition is satisfied.

Lemma 7.1. *For $r, s, l \geq 1$, extension (7.1) is central if and only if $r \geq l$.*

Proof. In the proof of Theorem 3.5 (which is included in Section 9), we show that $\Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+s})$ is generated by the matrices $\{A_{ij,k,r}\}$ and that $\Gamma(G_n(R), p^{r+s-l})/\Gamma(G_n(R), p^{r+s})$ is generated by the matrices $\{A_{ij,k,r+s-l}\}$. A straightforward calculation will verify that the following relation is satisfied, where $[g, h] = g^{-1}h^{-1}gh$ denotes the group commutator.

$$[A_{i_1 j_1, k_1, r+s-l}, A_{i_2 j_2, k_2, r}] = 1 + p^{2r+s-l} v_{k_1} v_{k_2} (e_{i_1 j_1} e_{i_2 j_2} - e_{i_2 j_2} e_{i_1 j_1}).$$

From this, we see that the commutator is trivial in Γ_r/Γ_{r+s} if and only if $2r+s-l \geq r + s$. $\qquad\square$

If $l = 1$ and $r \geq s-1$, one can check that extension (7.1) also satisfies the second condition mentioned above. Namely, defining a map $\theta : F[x, y] \longrightarrow \Gamma_r/\Gamma_{r+s}$ induces a morphism of extensions

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & F[x, y] & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \\
1 & \longrightarrow & \Gamma_{r+s-1}/\Gamma_{r+s} & \longrightarrow & \Gamma_r/\Gamma_{r+s} & \longrightarrow & \Gamma_r/\Gamma_{r+s-1} & \longrightarrow & 1.
\end{array}
$$

Let χ denote the generator of $H_2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}$. Working out the local coefficient system in the Lyndon-Hochschild-Serre spectral sequence for the extension

$$1 \longrightarrow K \longrightarrow F[x, y] \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 1,$$

one can check that $H_0(\mathbb{Z} \oplus \mathbb{Z}; H_1(K)) \cong \mathbb{Z}$, and this group is generated by the class of the element $[x, y] \in K$. Since it is known that $H_2(F[x, y]; \mathbb{Z}) = 0$, it must be the case that the differential $d^2_{2,0}$ is an isomorphism. In other words, $d^2_{2,0}(\chi) = [x, y]$.

The induced map θ_* on homology induces a map of spectral sequences associated to the two extensions. Using naturality,

$$d^2_{2,0}(\theta_*(\chi)) = \theta_*(d^2_{2,0}(\chi)) = \theta_*([x, y]) = [\theta_*(x), \theta_*(y)].$$

Thus, we have the following formula.

Theorem 7.2. *Suppose $r, s \geq 1$, $r \geq s - 1$, and a map $\theta : F[x,y] \longrightarrow \Gamma_r/\Gamma_{r+s}$ is given. Then θ induces a morphism of extensions as above, and*

$$d_{2,0}^2(\theta_*(\chi)) = [\theta_*(x), \theta_*(y)]$$

in the LHS spectral sequence for the extension

(7.2) $\qquad\qquad 1 \longrightarrow \Gamma_{r+s-1}/\Gamma_{r+s} \longrightarrow \Gamma_r/\Gamma_{r+s} \longrightarrow \Gamma_r/\Gamma_{r+s-1} \longrightarrow 1.$

Here, χ denotes the generator of $H_2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}$.

Notice that $[\theta_*(x), \theta_*(y)]$ can be calculated using the Lie algebra structure for $\mathrm{gr}_*(\Gamma(G_n(R), p))$ given in Theorem 4.7. This allows us to compute $d_{2,0}^2$ for many examples involving central extensions of congruence subgroups.

Remark. The reader should notice that the formula for the differential $d_{2,0}^2$ in Theorem 7.2 provides information that is dual, in a sense, to the extension class characterizing extension (7.2).

8. Proof of Theorem 3.2

We adapt the method of the proof of Theorem 3.1 as found in [11]. Suppose that $X \in \Gamma(G_n(R), p^r)$. We can write $X = 1 + p^r A$ where 1 denotes the identity matrix and $A \in \mathrm{Mat}(n, R)$. Define the map $\varphi_r : \Gamma(G_n(R), p^r) \longrightarrow \mathfrak{sl}_n(\mathbb{F}_p[V])$ by

$$\varphi_r(X) = A \mod p.$$

To show that the image of φ_r lies in $\mathfrak{sl}_n(\mathbb{F}_p[V])$, we make use of the fact that $\Gamma(G_n(R), p^r) \subseteq SL(n, R)$. Thus,

$$\begin{aligned}
1 &= \det X \\
&= \det(1 + p^r A) \\
&\equiv (1 + p^r \operatorname{trace}(A)) \mod p^{r+1}.
\end{aligned}$$

Thus, it must be the case that $\operatorname{trace}(A) \equiv 0 \mod p$.

One can check that φ_r is a homomorphism directly:

$$\begin{aligned}
\varphi_r((1 + p^r A)(1 + p^r B)) &= \varphi_r(1 + p^r(A + B + p^r AB)) \\
&= (A + B + p^r AB) \mod p \\
&\equiv (A + B) \mod p \\
&= \varphi_r(1 + p^r A) + \varphi_r(1 + p^r B).
\end{aligned}$$

To see that φ_r is surjective, we define a basis for $\mathfrak{sl}_n(\mathbb{F}_p[V])$ and show that φ_r surjects onto this basis. In the proof of Corollary 3.3, we defined e_{ij} to be the $n \times n$ matrix with a 1 in the (i,j) position and zeros elsewhere. A quick check will verify that a basis for $\mathfrak{sl}_n(\mathbb{F}_p[V])$ is given by $\{v_k e_{ij}\}_{i \neq j} \cup \{v_k(e_{ii} - e_{nn})\}_{i=1}^{n-1}$ for $k \in I$. Further,

$$\varphi_r(1 + p^r v_k e_{ij}) = v_k e_{ij}$$

and

$$\varphi_r(1 + p^r v_k(e_{ii} + e_{in} - e_{ni} - e_{nn})) = v_k(e_{ii} + e_{in} - e_{ni} - e_{nn})$$

so that φ_r hits all of the basis elements in $\mathfrak{sl}_n(\mathbb{F}_p[V])$.

Finally, notice that if $X \in \Gamma(G_n(R), p^{r+1})$, we can write $X = 1 + p^{r+1}A = 1 + p^r(pA)$ for some $A \in \mathrm{Mat}(n, R)$. So $\varphi_r(X) = 0$. Conversely, if $\varphi_r(1 + p^r A) = 0$, it must be the case that $A = pB$ for some $B \in \mathrm{Mat}(n, R)$. Thus, $1 + p^r A =$

$1 + p^{r+1}B \in \Gamma(G_n(R), p^{r+1})$. So $\ker(\varphi_r) = \Gamma(G_n(R), p^{r+1})$. This completes the proof of the theorem.

9. Proof of Theorem 3.5

We proceed by induction on s. The case $s = 1$ is Lemma 3.7. Suppose that $s-1$ and smaller cases have been proved. Consider the group extension

$$1 \longrightarrow \Gamma_{r+s-1}/\Gamma_{r+s} \longrightarrow \Gamma_r/\Gamma_{r+s} \longrightarrow \Gamma_r/\Gamma_{r+s-1} \longrightarrow 1,$$

where we again use the convention that $\Gamma_l = \Gamma(G_n(R), p^l)$. By Lemma 3.7,

$$\Gamma_{r+s-1}/\Gamma_{r+s} \cong \bigoplus_{n^2-1} \mathbb{Z}/p\mathbb{Z}[V],$$

so that $|\Gamma_{r+s-1}/\Gamma_{r+s}| = p^{(n^2-1)\cdot|V|}$. By induction,

$$\Gamma_r/\Gamma_{r+s-1} \cong \bigoplus_{n^2-1} \mathbb{Z}/p^{s-1}\mathbb{Z}[V].$$

This implies that $|\Gamma_r/\Gamma_{r+s}| = p^{(n^2-1)\cdot|V|} \cdot p^{(s-1)(n^2-1)\cdot|V|} = p^{s(n^2-1)\cdot|V|}$.

Any element of $\bigoplus_{n^2-1} \mathbb{Z}/p^s\mathbb{Z}[V]$ can be written as

$$(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{n,n-1})$$

where $a_{ij} \in \mathbb{Z}/p^s\mathbb{Z}[V]$. Let $\overline{e}_{ij} \in \bigoplus_{n^2-1} \mathbb{Z}/p^s\mathbb{Z}[V]$ be the element with a 1 in the the (i,j) position and zeros elsewhere. Then $\{v_k\overline{e}_{ij}\}$ for $1 \le i,j \le n$ and $i+j < 2n$ is a basis for $\bigoplus_{n^2-1} \mathbb{Z}/p^s\mathbb{Z}[V]$. Consider the map

$$\Theta_{r,s} : \bigoplus_{n^2-1} \mathbb{Z}/p^s\mathbb{Z}[V] \to \Gamma_r/\Gamma_{r+s}$$

that sends $v_k\overline{e}_{ij} \mapsto A_{ij,k,r}$, where the family $\{A_{ij,k,r}\}$ are the generators of Γ_r/Γ_{r+1} as defined in Corollary 3.8.

Claim. The group Γ_r/Γ_{r+s} is also generated by the matrices $\{A_{ij,k,r}\}$.

Proof of Claim. We again proceed by induction on s. The case $s = 1$ is clear. Suppose that $s - 1$ and smaller cases are satisfied. By induction, Γ_r/Γ_{r+s-1} is generated by the matrices $\{A_{ij,k,r}\}$. Using Corollary 3.8, $\Gamma_{r+s-1}/\Gamma_{r+s}$ is generated by the matrices $\{A_{ij,k,r+s-1}\}$. But for each i, j, and k,

$$A_{ij,k,r+s-1} = (\psi^p_{r+s-2} \circ \psi^p_{r+s-3} \circ \cdots \circ \psi^p_{r+1} \circ \psi^p_r)(A_{ij,k,r}) = A^{p^{s-1}}_{ij,k,r}.$$

Thus, Γ_r/Γ_{r+s} is generated by the matrices $\{A_{ij,k,r}\}$, which completes the proof of the claim.

Once we have this, the reader can check that since $r \ge s \ge 2$, the generators of Γ_r/Γ_{r+s} commute, so that Γ_r/Γ_{r+s} is abelian. So $\Theta_{r,s}$ is a homomorphism of abelian groups, which is clearly surjective since Γ_r/Γ_{r+s} is generated by the matrices $\{A_{ij,k,r}\}$.

Finally, since $|\bigoplus_{n^2-1} \mathbb{Z}/p^s\mathbb{Z}[V]| = p^{s(n^2-1)\cdot|V|} = |\Gamma_r/\Gamma_{r+s}|$, it must also be the case that $\Theta_{r,s}$ is injective, making it an isomorphism. This completes the proof of the theorem.

10. Proof of Theorem 2.4

Recall the map

$$\varphi_r : \Gamma(G_n(R), p^r) \longrightarrow \mathfrak{sl}_n(\mathbb{F}_p[V])$$

defined in Section 8. This induces a well-defined map on the filtration quotients

$$\overline{\varphi}_r : \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}) \longrightarrow \mathfrak{sl}_n(\mathbb{F}_p[V]).$$

Define a map

$$\overline{\varphi}_r \otimes t^r : \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}) \longrightarrow \mathfrak{sl}_n(\mathbb{F}_p[V]) \otimes_{\mathbb{F}_p} I$$

by

$$(\overline{\varphi}_r \otimes t^r)(X_r) = \overline{\varphi}_r(X_r) \otimes t^r$$

for $X_r \in \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1})$. We can sum these maps, in a sense, to obtain a map

$$\varphi : \mathrm{gr}_*(\Gamma(G_n(R), p)) \longrightarrow \mathfrak{sl}_n(\mathbb{F}_p[V]) \otimes_{\mathbb{F}_p} I$$

given by

$$\varphi(X) = \sum_{r \geq 1} (\overline{\varphi}_r \otimes t^r)(X_r).$$

Here, we write $X = (X_1, X_2, \ldots)$ since

$$\mathrm{gr}_*(\Gamma(G_n(R), p)) = \bigoplus_{r \geq 1} \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}).$$

To check that φ is surjective, we define a basis for $\mathfrak{sl}_n(\mathbb{F}_p[V]) \otimes_{\mathbb{F}_p} I$ and show that φ surjects onto this basis. In Section 8, we defined a basis for $\mathfrak{sl}_n(\mathbb{F}_p[V])$ to be $\{v_k e_{ij}\}_{i \neq j} \cup \{v_k(e_{ii} - e_{nn})\}_{i=1}^{n-1}$ for $k \in I$. The reader can check that $\{v_k e_{ij} \otimes t^r\}_{i \neq j} \cup \{v_k(e_{ii} - e_{nn}) \otimes t^r\}_{i=1}^{n-1}$ for $k \in I$ and $r \geq 1$ is a basis for $\mathfrak{sl}_n(\mathbb{F}_p[V]) \otimes_{\mathbb{F}_p} I$. It's clear that

$$\varphi(1 + p^r v_k e_{ij}) = v_k e_{ij} \otimes t^r.$$

Furthermore,

$$\varphi(1 + p^r v_k(e_{ii} + e_{in} - e_{ni} - e_{nn})) = v_k(e_{ii} + e_{in} - e_{ni} - e_{nn}) \otimes t^r,$$

so that φ hits all the basis elements in $\mathfrak{sl}_n(\mathbb{F}_p[V]) \otimes_{\mathbb{F}_p} I$.

Since $\overline{\varphi}_r$ was injective for all $r \geq 1$ (see the proof of Theorem 3.2 in Section 8), it must be the case that φ is injective as well.

Next, we check that φ is a homomorphism. Suppose $X, Y \in \mathrm{gr}_*(\Gamma(G_n(R), p))$. Write $X = (X_1, X_2, \ldots)$ and $Y = (Y_1, Y_2, \ldots)$, where

$$X_r, Y_r \in \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1})$$

for $r \geq 1$. Since each $\overline{\varphi}_r$ is a homomorphism, we have the following:

$$\varphi(X + Y) = \sum_{r \geq 1} (\overline{\varphi}_r \otimes t^r)(X_r Y_r)$$

$$= \sum_{r \geq 1} (\overline{\varphi}_r(X_r Y_r) \otimes t^r)$$

$$= \sum_{r \geq 1} ([\overline{\varphi}_r(X_r) + \overline{\varphi}_r(Y_r)] \otimes t^r)$$

$$= \sum_{r \geq 1} ([\overline{\varphi}_r(X_r) \otimes t^r] + [\overline{\varphi}_r(Y_r) \otimes t^r])$$

$$= \sum_{r \geq 1} (\overline{\varphi}_r(X_r) \otimes t^r) + \sum_{r \geq 1} (\overline{\varphi}_r(Y_r) \otimes t^r)$$

$$= \sum_{r \geq 1} (\overline{\varphi}_r \otimes t^r)(X_r) + \sum_{r \geq 1} (\overline{\varphi}_r \otimes t^r)(Y_r)$$

$$= \varphi(X) + \varphi(Y).$$

Thus, φ is a homomorphism.

Finally, we need to show that φ preserves the Lie bracket. Since $[X, Y] \in \text{gr}_*(\Gamma(G_n(R), p))$, we can write

$$[X, Y] = ([X, Y]_1, [X, Y]_2, \ldots),$$

where $[X, Y]_r \in \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1})$. Also, since

$$X_r, Y_r \in \Gamma(G_n(R), p^r)/\Gamma(G_n(R), p^{r+1}),$$

there exist $\widehat{X}_r, \widehat{Y}_r \in \text{Mat}(n, R)$ such that $X_r = 1 + p^r \widehat{X}_r$ and $Y_r = 1 + p^r \widehat{Y}_r$. We have the following:

$$\varphi([X, Y]) = \sum_{r \geq 1} (\overline{\varphi}_r \otimes t^r)([X, Y]_r)$$

$$= \sum_{r \geq 1} (\overline{\varphi}_r([X, Y]_r) \otimes t^r)$$

$$= \sum_{r \geq 1} (\overline{\varphi}_r(\sum_{i+j=r} [X_i, Y_j]) \otimes t^r)$$

$$= \sum_{r \geq 1} (\overline{\varphi}_r(\sum_{i+j=r} [1 + p^i \widehat{X}_i, 1 + p^j \widehat{Y}_j]) \otimes t^r)$$

$$= \sum_{r \geq 1} (\sum_{i+j=r} \overline{\varphi}_r([1 + p^i \widehat{X}_i, 1 + p^j \widehat{Y}_j]) \otimes t^r)$$

$$= \sum_{r \geq 1} (\sum_{i+j=r} ((\widehat{X}_i \widehat{Y}_j - \widehat{Y}_j \widehat{X}_i) \mod p) \otimes t^r)$$

$$= [\varphi(X), \varphi(Y)].$$

Thus,

$$\text{gr}_*(\Gamma(G_n(R), p)) \cong \mathfrak{sl}_n(\mathbb{F}_p[V]) \otimes_{\mathbb{F}_p} I$$

as Lie algebras. This completes the proof of the theorem.

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